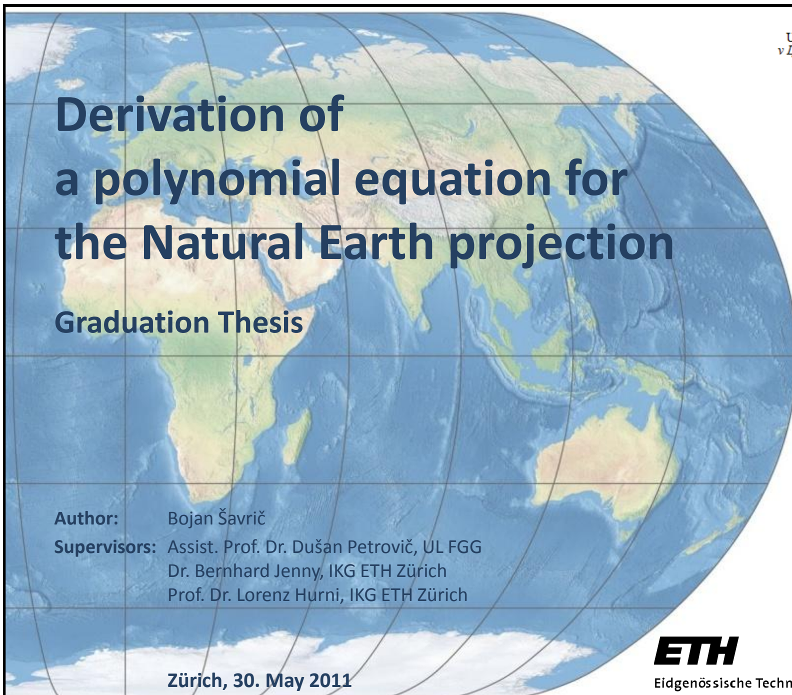

Derivation of a polynomial equation for the Natural Earth projection

Graduation Thesis

Author: Bojan Šavrič

Supervisors: Assist. Prof. Dr. Dušan Petrovič, UL FGG
Dr. Bernhard Jenny, IKG ETH Zürich
Prof. Dr. Lorenz Hurni, IKG ETH Zürich

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Univerza v Ljubljani
Fakulteta za gradbeništvo in geodezijo

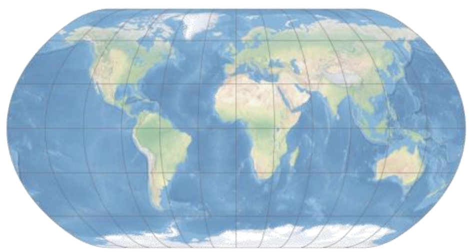
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

OVERVIEW

- The Natural Earth projection
- The problems and goals
- Analytical equations for the Robinson projection
- Used numerical methods
- Derivation of the equation
- Results (forward and inverse projection)
- Conclusion

THE NATURAL EARTH PROJECTION

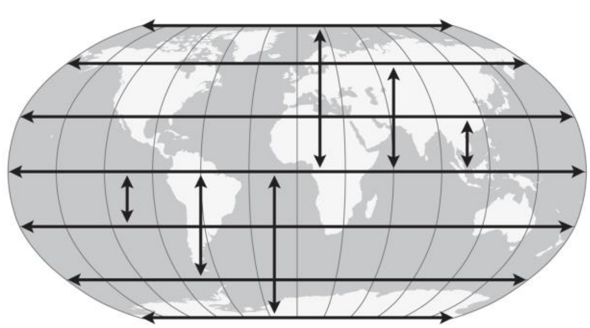
- Developed by Tom Patterson in 2007
- True pseudo-cylindrical projection
- Distinguishing characteristic: rounded corners



- Graphical design in Flex Projector
- Neither conformal nor equal area
- It exaggerates the size of high latitude areas

PROBLEMS OF THE PROJECTION (1)

- Defined by two tabular parameters for each five degrees of the latitude:
  - The length of the parallels – $l\varphi$
  - The distance of the parallels from the equator – $d\varphi$



PROBLEMS OF THE PROJECTION (2)

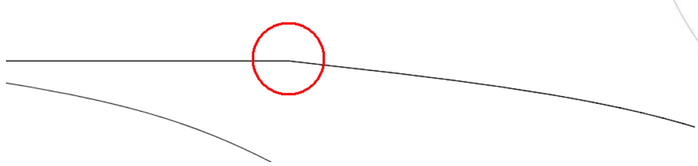
- Projection has no analytical equation
- Intermediate points are define with piece-wise cubic spline interpolation
- Implementation requires a lot of effort
- Only implement in Flex Projector

$$X = R \cdot s \cdot l_\varphi \cdot \lambda, \qquad l_\varphi \in [0,1]$$
$$Y = R \cdot s \cdot d_\varphi \cdot k \cdot \pi, \qquad d_\varphi \in [0,1]$$

Latitude	Length of Parallels	Distance of Parallels from Equator
0	1	0
5	0.9988	0.062
10	0.9953	0.124
15	0.9894	0.186
20	0.9811	0.248
25	0.9703	0.31
30	0.957	0.372
35	0.9409	0.434
40	0.9222	0.4958
45	0.9006	0.5571
50	0.8763	0.6176
55	0.8492	0.6769
60	0.8196	0.7346
65	0.7874	0.7903
70	0.7525	0.8435
75	0.716	0.8936
80	0.6754	0.9394
85	0.627	0.9761
90	0.563	1
Height / width		0.52
Scale		0.8707

PROBLEMS OF THE PROJECTION (3)

- Rounded corners →
  slight edge at the end of pole lines
- 5 degree spacing between latitude is not enough to completely smooth corners
- Wish by designer Tom Patterson

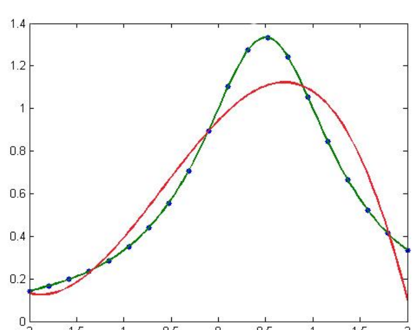


TWO GOALS OF THE THESIS

- An analytical expression
  - relating the spherical and Cartesian coordinates
  - small number of parameters
  - inverse function
- Improvement of the rounded corners where the border meridians meet the pole line

ANALYTICAL EQUATIONS FOR THE ROBINSON PROJECTION (1)

- Same graphical approach, same problems – not having normal analytical equations
- Two approaches for modeling the projection: *approximation* and *interpolation*



ANALYTICAL EQUATIONS FOR THE ROBINSON PROJECTION (2)

- Two approximations:
  - Canters and Decleir – 1989:
    (2 polynomial equations, 6 parameters)

$$X = R \cdot \lambda \cdot (A_0 + A_2 \cdot \varphi^2 + A_4 \cdot \varphi^4)$$
$$Y = R \cdot (A_1 \cdot \varphi + A_3 \cdot \varphi^3 + A_5 \cdot \varphi^5)$$

  - Beineke – 1991, 1995:
    (1 polynomial and 1 exponential equation, 8 parameters)

$$X = (d + e \cdot \varphi^2 + f \cdot \varphi^4 + g \cdot \varphi^6)\frac{\lambda}{\pi}$$
$$Y = a \cdot \varphi + b \cdot s \cdot |\varphi|^c$$

- Exponential equation is slower to evaluate than a polynomial

USED NUMERICAL METHODS

- For approximation:
  - Least squares adjustment (LSA)

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{l} + \boldsymbol{v}$$

  - LSA with additional constraints

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{l} + \boldsymbol{v}$$
$$\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{g}$$

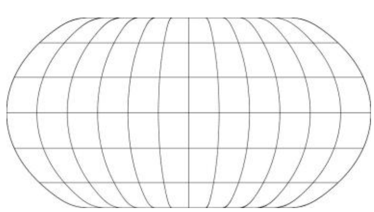
  - One tabular parameter value = one equation
- For inverse projection: Newton's method
  - From initial guess computes improved approximated roots (iterative procedure)

DERIVATION OF THE EQUATION

- Derivation contains six separated phases
- Approximation with the polynomials
- Two criteria:
  – Number of polynomial terms and multiplications to evaluate are minimized
  – Minimizing the absolute difference between the original and approximated projection
- Patterson’s graphical evaluation

Derivation of the polynomials

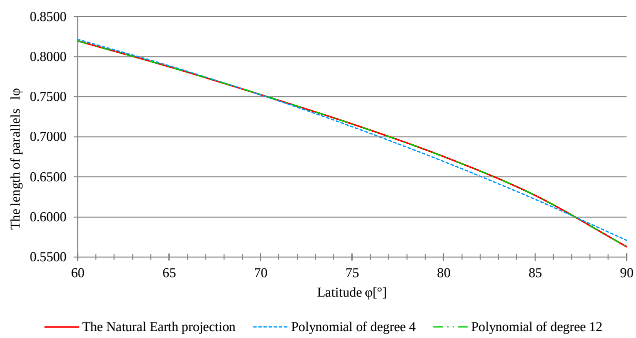
- Three characteristics of projection graticule were taken into account:



  – Projection is symmetric about the x and y-axis
  – It has straight but not equally spaced parallels
  – The parallels are equally divided by meridians
- For accelerated the computations, terms with small contribution were removed

The order of the polynomials

- Low degree – dissimilar curve to original one
- Higher degree better (increased order has curve closer to original projection – 12th degree is chosen)
- Maximal residuals and reference variance (LSA) are used for evaluation different degrees.



Adding constraints in the LSA

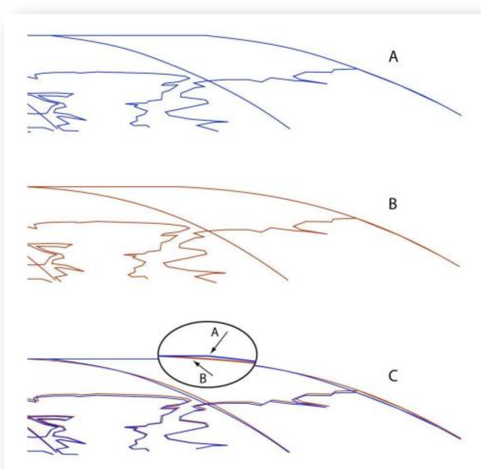
- Approximation → not the same size of graticule
- Width and height should be preserved
- Two additional constraints are applied:
  - The length of the parallels $l\varphi$ at 0 degrees stays 1
  - The distance between the equator and the pole line ($d\varphi = 90°$) stays 1
- LSA with additional constraints

Improving the rounded corners (1)

- Two additional measures are required:
  - Fixing the slope for the function of Y coordinate to 7 degrees at poles (graphically provides smooth corners, no edges)
  - Reducing the pole line from 0.563 to 0.550 (hides small undesired bulges caused by slope)
- Slope is applied as additional constraint
- Reducing the pole line is done before computing the polynomial coefficients

Improving the rounded corners (2)

- Result of improvement:



RESULTS (1)

- Improved Natural Earth projection with equations
- Forward polynomial expressions:

$$X = R \cdot \lambda \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12})$$
$$Y = R \cdot (B_1\varphi + B_2\varphi^3 + B_3\varphi^7 + B_4\varphi^9 + B_5\varphi^{11})$$

where:

X and Y are the projected coordinates,

φ and λ are the latitude and longitude in radians,

R is the radius of the generating globe,

A_1 to A_5 and B_1 to B_5 are coefficients given in table below

Coefficients for X function		Coefficients for Y function	
A_1	0.870700	B_1	1.007226
A_2	-0.131979	B_2	0.015085
A_3	-0.013791	B_3	-0.044475
A_4	0.003971	B_4	0.028874
A_5	-0.001529	B_5	-0.005916

RESULTS (2)

- Inverse projection (four steps)

(1) The initial guess for the unknown latitude: $\varphi_0 = Y \cdot R^{-1}$

(2) With the Newton's root finding algorithm improved latitude φ is calculated:

$$\varphi_{n+1} = \varphi_n - F(\varphi_n) \cdot (F'(\varphi_n))^{-1}$$

where:

$F(\varphi_n) = B_1\varphi_n + B_2{\varphi_n}^3 + B_3{\varphi_n}^7 + B_4{\varphi_n}^9 + B_5{\varphi_n}^{11} - Y \cdot R^{-1} = 0,$

$F'(\varphi_n)$ its derivative and $n = 0,1,2, \ldots, m$.

At step m the iteration stops if $|\varphi_{m+1} - \varphi_m| < \varepsilon$,

where ε is sufficiently small positive quantity,

typically close to the maximum precision of floating point arithmetic.

(3) The final latitude: $\varphi = \varphi_{m+1}$

(4) The final longitude:

$\lambda = X \cdot R^{-1} \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12})^{-1}$

COMPARISON OF BOTH GRATICULES

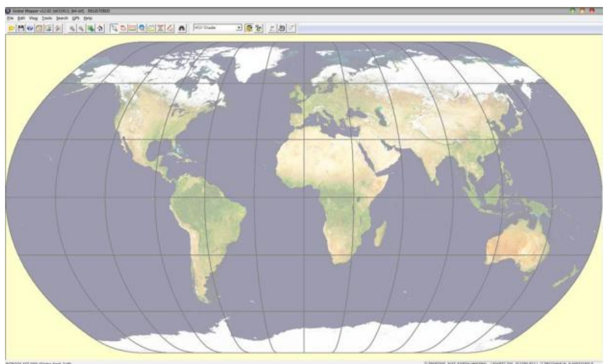
- Some deviations are present at corners (caused by smoothing the edges)
- The pole line is reduced by small amount
- The lengths of equator and central meridian are not changed
- Graticule at a scale of 1 : 5.000.000: pole line is reduced for almost 45 mm, other point has less than 2.5 mm of deviations
- Distortion values are almost identical

CONCLUSION (1)

- Polynomial equation for the projection:
  - Contains only 10 parameters,
  - Has the inverse projection,
  - Improves the graticule, (rounded corners are smooth completely)
  - Easy to compute and implement
- Both goals are reached
- Patterson recommends this polynomial equation as true analytical expression for the Natural Earth projection

CONCLUSION (2)

- Submitted paper to the Cartography and Geographic Information Science
- Global Mapper v12.02



THANK YOU FOR YOUR ATTENTION!